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**421. Proposed by E. H. MOORE, The University of Chicago.**

Given  $n$  continuous real-valued functions  $\phi_g(x)$  ( $g = 1, 2, \dots, n$ ) of the real variable  $x$  on the interval (01) and set  $\exp. \int_0^1 \phi_g(x) \phi_h(x) = w_{gh}$  ( $g, h = 1, 2, \dots, n$ ). Prove that the determinant  $|w_{gh}|$  of the matrix  $(w_{gh})$  is always  $\geq 0$  and that it is  $> 0$  if no two of the functions  $\phi_1, \dots, \phi_n$  are identically equal on (01).

SOLUTION BY C. F. GUMMER, Kingston, Ont.

(1) *Proof that  $|w_{gh}| \geq 0$ .*

$$w_{gh} = \lim_{m \rightarrow \infty} w_{gh}^{(m)}, \quad \text{where} \quad w_{gh}^{(m)} = 1/m \sum_{i=1}^m \phi_g(i/m) \phi_h(i/m);$$

and since  $|w_{gh}|$  is a continuous function of the  $w$ 's, it follows that

$$|w_{gh}| = |\lim w_{gh}^{(m)}| = \lim |w_{gh}^{(m)}|.$$

Now

$$|w_{gh}^{(m)}| = 1/m^n |\sum \phi_g(i/m) \phi_h(i/m)|,$$

which, by the rule for a minor of a product matrix, is equal to  $(1/m^n) \times$  the sum of the squares of the  $n$ th order determinants of the matrix  $(\phi_g(i/m))_{\substack{i=1, \dots, m \\ g=1, \dots, n}}$ .

$$\therefore |w_{gh}^{(m)}| \geq 0.$$

$$\therefore |w_{gh}| = \lim |w_{gh}^{(m)}| \geq 0.$$

(2) *The condition that  $|w_{gh}| = 0$ .*

Suppose  $|w_{gh}| = 0$ . Then there is a real linear relation

$$\sum_{g=1}^n a_g w_{gh} = 0 \quad (h = 1, 2, \dots, n). \quad (A)$$

$$\therefore \int_0^1 \{\sum a_g \phi_g(x)\} \phi_h(x) dx = 0 \quad (h = 1, \dots, n).$$

Multiplying by  $a_h$  and adding, we have

$$\int_0^1 \{\sum a_g \phi_g(x)\}^2 dx = 0.$$

Hence, the integrand being continuous,

$$\sum a_g \phi_g(x) = 0 \text{ on } (01). \quad (B)$$

Conversely, if (B) is true, so is (A), and  $|w_{gh}| = 0$ .  $\therefore$  the necessary and sufficient condition for the vanishing of  $|w_{gh}|$  is that the  $\phi$ 's be linearly dependent, and the problem as stated is incorrect. (Consider for example the case  $\phi_1 = 1, \phi_2 = x, \phi_3 = 1 + x$ .)

**MECHANICS.****330. Proposed by PAUL CAPRON, U. S. Naval Academy.**

A Barker's Mill operates under a head of  $h$  ft.; the linear speed of the orifices is  $u$  (f/s), the speed of the water relative to the orifices is  $v$  (f/s), the coefficient of discharge is  $c$ , so that  $v^2 = c^2(2gh + u^2)$ . Given that the work done by the water on the mill is  $u/g(v - u)$  ft. lbs. per sec. per lb. of water used, find the values of  $u$  and  $v$  such that the water-power may be most economically used, and find what part of the power is so used.

SOLUTION BY THE PROPOSER.

It is required to make  $f(u) = u(c\sqrt{k^2 + u^2} - u)$  a maximum. ( $k^2 = 2gh$ ).

$$f'(u) = \frac{cu^2}{\sqrt{k^2 + u^2}} + c\sqrt{k^2 + u^2} - 2u = 0$$

when  $c(u^2 + k^2 + u^2) = 2u\sqrt{k^2 + u^2}$ , or  $4(1 - c^2)(u^4 + k^2u^2) = c^2k^4$ .

Whence,

$$u^2 = \frac{k^2}{2} \left( \frac{1 - \sqrt{1 - c^2}}{\sqrt{1 - c^2}} \right),$$

and

$$v^2 = c^2(k^2 + u^2) = \frac{c^2 k^2}{2} \left( \frac{1 + \sqrt{1 - c^2}}{\sqrt{1 - c^2}} \right).$$

Hence,

$$u = \frac{k}{2\sqrt[4]{1 - c^2}} (\sqrt{1 + c} - \sqrt{1 - c}), \quad v = \frac{ck}{2\sqrt[4]{1 - c^2}} (\sqrt{1 + c} + \sqrt{1 - c}),$$

and

$$u(v - u) = \frac{k^2}{2} (1 - \sqrt{1 - c^2}).$$

The available work is  $h$  ft. lbs. per sec. per lb. of water used; the work utilized is

$$\frac{u}{g} (v - u) = h(1 - \sqrt{1 - c^2});$$

the proportion used is  $(1 - \sqrt{1 - c^2})$ .

If we let  $c = \sin \alpha$ , we have  $u^2 = gh \tan \alpha \tan \alpha/2$ ,  $v = 2u \cos^2 \alpha/2$ , and efficiency = vers  $\alpha$ .

### 331. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.

A cyclist is riding due west at a speed of 12 miles per hr., and the wind is at the same time blowing from the southeast with a speed of  $5\frac{1}{2}$  miles per hour. If the cyclist carries a small flag, in what direction will this flag fly? At what speed would the cyclist need to ride if the flag is to fly due north?

#### SOLUTION BY B. J. BROWN, Victor, Colo.

The component of the wind north is  $11\sqrt{2}/4$  miles per hr., while that west is  $11\sqrt{2}/4$  miles per hr. The resistance offered to the machine traveling west is  $[12 - (11\sqrt{2}/4)]$  and the reaction is toward the east. Hence the forces affecting the flag are  $[12 - (11\sqrt{2}/4)]$  toward east, and  $11\sqrt{2}/4$  toward north and the direction  $\theta$ , which the resultant makes with the east-west line =  $\arctan 11\sqrt{2}/(48 - 11\sqrt{2}) = 25^\circ 36' 55''$  N. of E. In order for the flag to fly due N. the rider must travel W. at rate of  $11\sqrt{2}/4$  miles per hr. Then he counteracts the resistance E. and only the component north is effective.

Also solved by PAUL CAPRON, W. J. THOME, and G. W. HARTWELL.

#### NUMBER THEORY.

### 236. Proposed by V. M. SPUNAR, Chicago, Illinois.

Find integral values of  $x, y, z$ , such that

$$xy + z = \square, \quad yz + x = \square, \quad zx + y = \square.$$

#### NOTE BY G. H. LING, University of Saskatchewan.

The following proposition generalizes somewhat the solution of this problem which was given in the February, 1917, MONTHLY.

**THEOREM.** If (1)  $a$  is any integer, (2)  $N_1, N_2$  are integers such that  $N_1 \cdot N_2 = a^2 + 1$ , (3)  $x = N_1 - 1, y = N_2 - 1, z = N_1 + N_2 - 1 + 2a$ , then

$$(I) \quad xy + z = (a + 1)^2; \quad yz + x = (N_2 + a - 1)^2; \quad zx + y = (N_1 + a - 1)^2,$$

$$(II) \quad xy + x + y = a^2; \quad yz + y + z = (N_2 + a)^2; \quad zx + x + z = (N_1 + a)^2.$$

The solution given in the MONTHLY is the special case of this one in which

$$a = n^2 + n + 1.$$